Inventory model with Quadratic Demand, Weibull distribution and Generalized Pareto Decay

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Abstract

Inventory models play an important role in determining the optimal ordering and pricing policies. Economic production quantity models usually assume that the production is fixed and finite. Much work has been reported in literature regarding inventory models with finite or infinite replenishment. But in many practical situations the replenishment is governed by random factors like procurement, transportation, environmental condition, availability of raw material etc., hence, it is needed to develop inventory models with random replenishment. In this paper, an EPQ model for deteriorating items is developed and analyzed with the assumption that the replenishment is random and follows a Weibull distribution. It is further assumed that the life time of a commodity is random and follows a generalized Pareto distribution and demand is a function of a quadratic demand. Using the differential equations, the instantaneous state of inventory is derived. Assuming that shortages are allowed and fully backlogged the instantaneous state of on hand inventory is derived. With suitable cost considerations, the total cost function is obtained. By minimizing the total cost function, the optimal ordering policies are derived. The models are studied and presented with a suitable numerical value of all parameters in standard form.

Keywords: Random replenishment; Generalized Pareto decay; Quadratic Demand; EPQ model; Weibull distribution.

1. Introduction

Several inventory models have been developed and analyzed to study various inventory systems. Much work has been reported in literature regarding Economic Production Quantity (EPQ) models during the last two decades. The EPQ models are also a particular case of inventory models. The major constituent components of the EPQ models are 1) Demand 2) production (Production) (Replenishment) and 3) Life time of the commodity. Several EPQ models have been developed and analyzed with various assumptions on demand pattern and life time of the commodity. In general, it is customary to consider that the replenishment is random in production inventory models.

Several researchers have developed various inventory models with stock dependent demand. In inventory system, different types of demand generally considered are constant demand, time dependent demand, price dependent demand, probabilistic demand and stock dependent demand. It is a common belief that large piles of goods displayed in super markets will lead the customer to buy more (Levin et. al [1]). Silver and Peterson [2] have also noted that the sales at retail level tend to be proportional to the inventory displayed. Since then, researchers have made attempts to investigate inventory models assuming a functional form between the demand rate and on-hand inventory. Gupta and Vrat [3] suggested inventory models with variable rates of demand, Baker and Uber [4] formulated a deterministic inventory system with stock dependent demand rate, Mandal and Phaujdar [5] discussed an Inventory Model for Deteriorating Items and Stock Dependent Consumption Rate. Datta and Pal [6] presented a note on an inventory level dependent demand rate, Goh [7] developed an EOQ models with general demand and holding cost function, Ray and Chaudhuri [8] presented an EOQ model under inflation and time discounting allowing shortages, Beltran and Krass [9] discussed about dynamic lot sizing with returning items and disposals. Raman Patel and Reena [10] discussed an inventory model for Weibull deteriorating items with stock dependent demand, time varying holding cost and variable selling price. M.S.Reddy and Venkateswarlu. [11] and others have developed inventory models for deteriorating items with stock dependent demand. In all these models they assumed that the replenishment is instantaneous or having fixed finite rate, except Sridevi, et al. [12] that developed and analyzed an inventory model with the assumption that the rate of production is random and follows a Weibull distribution.

Another important consideration for developing the EPQ models for deteriorating items is the life time of the commodity. For items like food, processing the life time of the commodity is random and follows a generalized Pareto distribution. Very little work has been reported in the literature regarding EPQ models for deteriorating items with random replenishment and generalized Pareto decay having stock dependent demand, even though Laksjmana Rao and Srinivasa Rao [13] deriving the inventory model for deteriorating items with Weibull replenishment and generalized pareto decay having demand as function of on hand inventory. Hence, in this paper, we develop and analyze an economic production quantity model for deteriorating items with Weibull rate of replenishment and generalized Pareto decay having quadratic demand is a function of on hand inventory. The generalized Pareto distribution is capable of characterizing the life time of the commodities which have a minimum period to start deterioration, and the rate of deterioration is inversely proportionate to time.

Using the differential equations, the instantaneous state of inventory is derived. With suitable cost considerations, the total cost function is derived. By minimizing the total cost function, the optimal ordering policies are derived. The models are studied and presented with a suitable numerical value of all parameters in standard form.

2. Notations and assumptions

2.1 Notations

The following assumptions are made for developing the model.

i) The Demand rate D(t) at time t is assumed to be $d(t) = a + bt + ct^2, a \geq 0, b \neq 0, c \neq 0$. where a is the initial rate of demand, b is the initial rate of change of the demand and c is the acceleration of demand rate.

ii) The replenishment is finite and follows a two parameter Weibull distribution with probability density function

$$f(t) = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}; \quad \alpha > 0, \beta > 0, t > 0$$

Therefore, the instantaneous rate of replenishment is

$$k(t) = \frac{f(t)}{1 - F(t)} = \alpha\beta t^{\beta-1}; \quad \alpha > 0, \beta > 0, t > 0$$

iii) Lead time is zero

iv) Cycle length, T is known and fixed
v) Shortages are allowed and fully backlogged
vi) A deteriorated unit is lost
vii) The deterioration of the item is random and follows a generalized Pareto distribution. Then the instantaneous rate of deterioration is

$$h(t)=\frac{1}{x-\gamma t},\ 0<t<\frac{x}{\gamma}$$

2.2. Assumptions

The following notations are used for developing the model.

Q: Ordering quantity in one cycle

A: Ordering cost

C: Cost per unit

h: Inventory holding cost per unit per unit time

π: Shortages cost per unit per unit time

s: Selling price per unit

3. **Inventory model with shortages**

Consider an inventory system in which the stock level is zero at time $t = 0$. The Stock level increases during the period $(0, t_1)$, due to excess of replenishment after fulfilling the demand and deterioration. The replenishment stops at time t_1 when the stock level reaches S. The inventory decreases gradually due to demand and deterioration in the interval (t_1, t_2). At time t_2, the inventory reaches zero and back orders accumulate during the period (t_2, t_3). At time t_3, the replenishment starts again and fulfils the backlog after satisfying the demand. During (t_3, T), the back orders are fulfilled and inventory level reaches zero at the end of the cycle T. The Schematic diagram representing the instantaneous state of inventory is given in Figure 1.

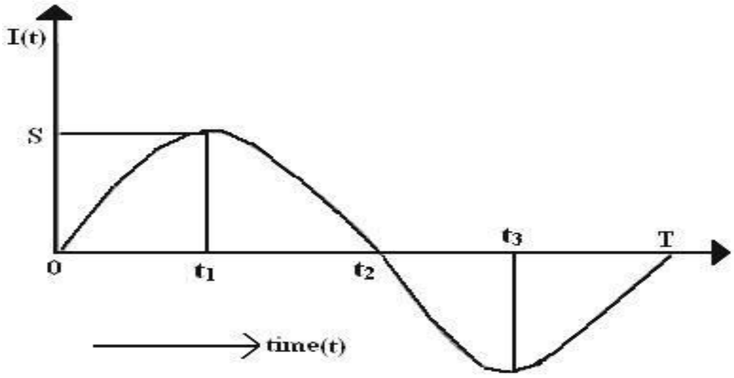

Figure 1. Schematic diagram representing the inventory level.

The differential equations governing the system in the cycle time [0, T] are:

$$\frac{d}{dt}I(t)+h(t)I(t)=\alpha\beta t^{\beta-1}-(a+bt+ct^2); \qquad 0\le t\le t_1 \tag{1}$$

$$\frac{d}{dt}I(t)+h(t)I(t)=-(a+bt+ct^2); \qquad t_1\le t\le t_2 \tag{2}$$

$$\frac{d}{dt}I(t)=-(a+bt+ct^2); \qquad t_2\le t\le t_3 \tag{3}$$

$$\frac{d}{dt}I(t)=\alpha\beta t^{\beta-1}-(a+bt+ct^2); \qquad t_3\le t\le T \tag{4}$$

The solutions of differential equations from the equations (1) – (4) by using the initial conditions, *I(0) = 0, I(t₁) = S, I(t₂) = 0* and *I(T) = 0*, the on hand inventory at time '*t*' is obtained as follows:

$$I(t)=S\left(\frac{x-\gamma t}{x-\gamma t_1}\right)^{1/\gamma}+(x-\gamma t)^{1/\gamma}\begin{pmatrix}\int(\alpha\beta t^{\beta-1}-a-bt-ct^2)(x-\gamma t)^{-1/\gamma}dt\\-\int(\alpha\beta t_1^{\beta-1}-a-bt_1-ct_1^2)(x-\gamma t_1)^{-1/\gamma}dt\end{pmatrix},\ 0\le t\le t_1 \quad (5)$$

$$I(t)=S\left(\frac{x-\gamma t}{x-\gamma t_1}\right)^{1/\gamma}+(x-\gamma t)^{1/\gamma}\begin{pmatrix}\int(a+bt_1+ct_1^2)(x-\gamma t_1)^{-1/\gamma}dt\\-\int(a+bt+ct^2)(x-\gamma t)^{-1/\gamma}dt\end{pmatrix},\ t_1\le t\le t_2 \quad (6)$$

$$\therefore I(t)=a(t_2-t)+b\left(\frac{t_2^2-t^2}{2}\right)+c\left(\frac{t_2^3-t^3}{3}\right),\qquad t_2\le t\le t_3 \quad (7)$$

$$I(t)=a(T-t)+b\left(\frac{T^2-t^2}{2}\right)+c\left(\frac{T^3-t^3}{3}\right)-\alpha\left(T^\beta-t^\beta\right),\qquad t_3\le t\le T \quad (8)$$

Stock loss due to deterioration in the interval (0, t) is

$$L(t)=\int_0^t k(t)dt-\int_0^t d(t)dt-I(t),\qquad 0\le t\le t_2$$

This implies

$$L(t)=\begin{cases}\alpha t^\beta-\left(at+\frac{bt^2}{2}+\frac{ct^3}{3}\right)-S\left(\frac{x-\gamma t}{x-\gamma t_1}\right)^{1/\gamma}\\-(x-\gamma t)^{1/\gamma}\begin{pmatrix}\int(\alpha\beta t^{\beta-1}-a-bt-ct^2)(x-\gamma t)^{-1/\gamma}dt\\-\int(\alpha\beta t_1^{\beta-1}-a-bt_1-ct_1^2)(x-\gamma t_1)^{-1/\gamma}dt\end{pmatrix}; & 0\le t\le t_1\\ \alpha t_1^\beta-\left(at_1+\frac{bt_1^2}{2}+\frac{ct_1^3}{3}\right)-S\left(\frac{x-\gamma t}{x-\gamma t_1}\right)^{1/\gamma}\\-(x-\gamma t)^{1/\gamma}\begin{pmatrix}\int(a+bt_1+ct_1^2)(x-\gamma t_1)^{-1/\gamma}dt\\-\int(a+bt+ct^2)(x-\gamma t)^{-1/\gamma}dt\end{pmatrix}; & t_1\le t\le t_2\end{cases} \quad (9)$$

Stock loss due to deterioration in the cycle of length T is

$$L(T)=\alpha t_1^\beta-\left(at+\frac{bt^2}{2}+\frac{ct^3}{3}\right) \quad (10)$$

Ordering quantity Q in the cycle of length T is

$$Q=\int_o^{t_1}k(t)dt+\int_{t_3}^{T}k(t)dt$$

$$\therefore Q=\alpha t_1^\beta+\alpha T^\beta-\alpha t_3^\beta=\alpha\left(t_1^\beta+T^\beta-t_3^\beta\right) \quad (11)$$

From equation (5) using the condition I(0) = 0, we obtain the value of 'S' as

$$\therefore S=(x-\gamma t_1)^{1/\gamma}\left(atx^{-1/\gamma}+(x-\gamma t_1)^{1/\gamma}\int(\alpha\beta t_1^{\beta-1}-a-bt_1-ct_1^2)(x-\gamma t_1)^{-1/\gamma}dt\right)$$

Let $TC(t_1,t_2,t_3)$ be the total cost per unit time. Since the total cost is the sum of the et-up cost, cost of the units, the inventory holding cost and shortage cost, the total cost per unit time becomes

$$TC(t_1,t_2,t_3)=\frac{A}{T}+\frac{CQ}{T}+\frac{h}{T}\left(\int_0^{t_1} I(t)dt+\int_{t_1}^{t_2} I(t)dt\right)+\frac{\pi}{T}\left(\int_{t_2}^{t_3} -I(t)dt+\int_{t_3}^{T} -I(t)dt\right) \quad (12)$$

On Substituting the values of I(t) and Q in equation (12), we can obtain $TC(t_1,t_2,t_3)$ as

$$TC(t_1,t_2,t_3)=\frac{A}{T}+\frac{C}{T}\left(\alpha\left(t_1^{\beta}+T^{\beta}-t_3^{\beta}\right)\right)$$

$$+\frac{h}{T}\left(\begin{array}{l}\int_0^{t_1}\left(S\left(\frac{x-\gamma t}{x-\gamma t_1}\right)^{1/\gamma}+(x-\gamma t)^{1/\gamma}\left(\begin{array}{l}\int(\alpha\beta t^{\beta-1}-a-bt-ct^2)(x-\gamma t)^{-1/\gamma}dt\\-\int(\alpha\beta_1^{\beta-1}-a-bt_1-ct_1^2)(x-\gamma t_1)^{-1/\gamma}dt\end{array}\right)\right)dt\\+\int_{t_1}^{t_2}\left(S\left(\frac{x-\gamma t}{x-\gamma t_1}\right)^{1/\gamma}+(x-\gamma t)^{1/\gamma}\left(\begin{array}{l}\int(a+bt_1+ct_1^2)(x-\gamma t_1)^{-1/\gamma}dt\\-\int(a+bt+ct^2)(x-\gamma t)^{-1/\gamma}dt\end{array}\right)\right)dt\end{array}\right)$$

$$-\frac{\pi}{T}\left(\begin{array}{l}\int_{t_2}^{t_3}\left(a(t_2-t)+b\left(\frac{t_2^2-t^2}{2}\right)+c\left(\frac{t_2^3-t^3}{3}\right)\right)dt+\\\int_{t_3}^{T}\left(a(T-t)+b\left(\frac{T^2-t^2}{2}\right)+c\left(\frac{T^3-t^3}{3}\right)-\alpha\left(T^{\beta}-t^{\beta}\right)\right)dt\end{array}\right) \quad (13)$$

Differentiating equation (13) with respect to t_1, t_2 and t_3, we get

$$\frac{\partial TC}{\partial t_1},\frac{\partial TC}{\partial t_2} \text{ and } \frac{\partial TC}{\partial t_3}$$

The optimal values of t_1, t_2 and t_3 can be obtained by solving the following equations in order to minimize the total cost TC (t_1, t_2 , t_3) per unit time:

$$\frac{\partial TC}{\partial t_1}=0,\ \frac{\partial TC}{\partial t_2}=0 \text{ and } \frac{\partial TC}{\partial t_3}=0$$

provided the determinant of principal minor of hessian matrix (H-Matrix) of TC (t_1, t_2 , t_3) is positive definite. i.e., det (H_1) >0, det (H_2) >0, det (H_3) > 0, where H_1, H_2, H_3 are the principal minors of the H-Matrix.

The Hessian matrix of the total cost TC (t_1, t_2 , t_3) is given by

$$H=\begin{bmatrix}\frac{\partial^2 TC}{\partial t_1^2} & \frac{\partial^2 TC}{\partial t_1\partial t_2} & \frac{\partial^2 TC}{\partial t_1\partial t_3}\\ \frac{\partial^2 TC}{\partial t_2\partial t_1} & \frac{\partial^2 TC}{\partial t_2^2} & \frac{\partial^2 TC}{\partial t_2\partial t_3}\\ \frac{\partial^2 TC}{\partial t_3\partial t_1} & \frac{\partial^2 TC}{\partial t_3\partial t_2} & \frac{\partial^2 TC}{\partial t_3{}^2}\end{bmatrix}$$

4. Numerical Illustrations

To illustrate the model, an inventory system is considered with the following parameter in proper units:

$A=250 \quad a=25 \quad b=15 \quad c=0.05 \quad T=12 \quad C=20$

$\alpha=2 \quad \beta=0.1 \quad \gamma=0.05 \quad \pi=5 \quad h=0.4 \quad X=0.5$

The output of the program by using the MATHCAD is $t_1=0.2936$, $t_2=3.7884$, $t_3=10.1065$ and $TC(t_1,t_2,t_3)=156.6$ i.e. the value of t_1 at which the inventory level become zero is 0.2936 unit time, shortage period is 1.5414 unit time and observed that the deterioration parameters and replenishment parameters have influence on the optimal replenishment times, ordering quantity and total cost.

We have compared this solution with time dependent linear demand i.e., c = 0. In the above numerical example, we have taken c = 0. Thus, we obtain $t_1 = 0.2898$, $t_2 = 3.8151$, $t_3 = 9.1254$ and $TC(t_1, t_2, t_3) = 148.8$. It is also observed that the deterioration parameters and replenishment parameters have influence on the optimal replenishment times, ordering quantity and total cost. (i.e., time dependent linear demand model and time dependent quadratic demand model). Thus, it may be concluded that the two models exhibit similar characteristics.

5. Conclusions

In this paper, we developed a deterministic inventory model for deteriorating items with shortages using preservation technology for deteriorating items. The deterministic demand rate was assumed to be a quadratic function of time. The holding cost was assumed as linear function of time. It was noted that the cost of preservation technology increases when the rate of deterioration increases from 0.01 to 0.05. It was also observed that the use of preservation technology to control the deterioration rate is completely independent of time dependent demand patterns.

6. References

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